Chapter 4
Phi-Regret and Multicalibration

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The black-box reduction from Φ-regret to online learning due to Gordon, Greenwald and Marks [GGM08] centers around the algorithmic primitive of fixed points. This chapter introduces a different route to the same destination that passes through forecasting [FP26].

At a high level, the moral of this section is the following, also depicted in Figure 1.

> One can construct a Φ-regret minimizer starting from a powerful enough *multicalibrated forecaster* of the upcoming utility, and best respond to such forecast. In turn, the task of constructing such a forecaster admits a reduction to online learning in the style of Gordon-Greenwald-Marks, with expected variational inequalities replacing expected fixed points as the nonlinear optimization primitive.

As we will show towards the end of the chapter, this forecasting-based reduction comes with algorithmic benefits that forego the need for complicated semiseparation (*cf.* Chapter 3, Section 3.2). We note however that it is not known whether a forecasting-based approach can yield fast *offline* algorithms for Φ-equilibrium computation (*cf.* Chapter 2, Section 2.3).

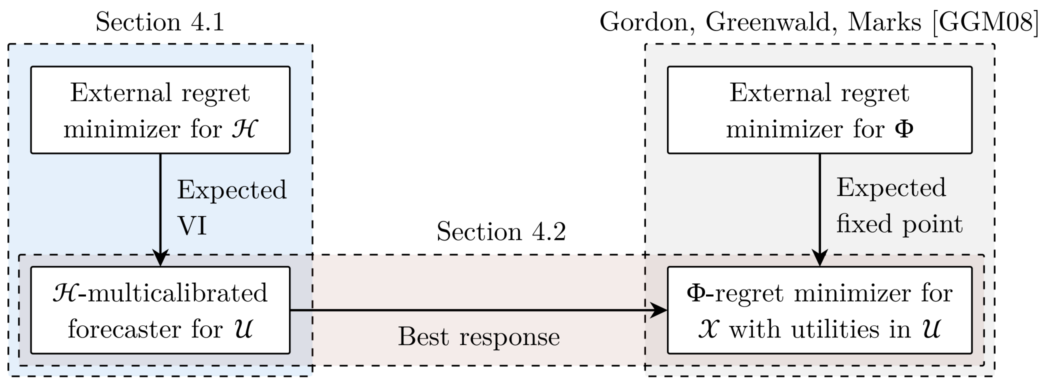

Figure 1: The two forecasting reductions and their relation to the Gordon-Greenwald-Marks framework.

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4.1 A Gordon-Greenwald-Marks result for multicalibration

We start with online multicalibration. In every round t, Nature reveals a context $\boldsymbol{c}^{(t)} \in \mathcal{C}$. The learner outputs a distribution $D^{(t)}$ over forecasts $\boldsymbol{p}^{(t)} \in \mathcal{U} \subseteq \mathbb{R}^d$, and Nature then reveals the true utility vector $\boldsymbol{u}^{(t)} \in \mathcal{U}$. Given test $h : \mathcal{C} \times \mathcal{U} \to \mathbb{R}^d$, the calibration error against h is

$$\text{MC-Err}^{(T)}(h) := \sum_{t=1}^{T} \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h\left(\boldsymbol{c}^{(t)}, \boldsymbol{p}^{(t)}\right), \boldsymbol{u}^{(t)} - \boldsymbol{p}^{(t)} \right\rangle \right]. \tag{1}$$

The forecasts are $\mathcal{H}$-multicalibrated if $\text{MC-Err}^{(T)}(h) = o(T)$ for every $h \in \mathcal{H}$.

As we show in Theorem 4.2, multicalibration admits a black-box reduction to online learning in the style of Gordon, Greenwald and Marks [GGM08]. However, in the case of multicalibration the nonlinear primitive is *not* fixed points, but rather *expected variational inequalities*, as defined next.[1]

Definition 4.1 (Expected variational inequality)**.** Let $S : \mathcal{U} \to \mathbb{R}^d$ be an operator and let $\varepsilon > 0$. An ε-*solution* to the expected variational inequality induced by S is a distribution D over $\mathcal{U}$ such that

$$\mathop{\mathbb{E}}_{\boldsymbol{p} \sim D} [\langle S(\boldsymbol{p}), \boldsymbol{u} - \boldsymbol{p} \rangle] \le \varepsilon \quad \forall \boldsymbol{u} \in \mathcal{U}. \tag{2}$$

Efficient algorithms for EVIs are known for general compact convex sets under mild oracle access assumptions [ZAT+25].

Armed with the EVI primitive, we can reduct multicalibration to an external-regret minimizer $R_{\mathcal{H}}$ whose decision set is the class of tests $\mathcal{H}$ as follows.

Algorithm: Multicalibration from external regret

Input: An external regret minimizer $R_{\mathcal{H}}$ for the test set $\mathcal{H}$

function `NextForecast`$(\boldsymbol{c}^{(t)})$:

- Set $h^{(t)} := R_{\mathcal{H}}$.`NextStrategy()`
- Let $D^{(t)}$ solve the EVI with $S^{(t)}(\boldsymbol{p}) := h^{(t)}(\boldsymbol{c}^{(t)}, \boldsymbol{p})$:

$$\mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h^{(t)}\left(\boldsymbol{c}^{(t)}, \boldsymbol{p}^{(t)}\right), \boldsymbol{u} - \boldsymbol{p}^{(t)} \right\rangle \right] \le \varepsilon^{(t)} \quad \forall \boldsymbol{u} \in \mathcal{U}$$

- **return** $D^{(t)}$

function `ObserveUtility`$(\boldsymbol{u}^{(t)})$:

- Feed to $R_{\mathcal{H}}$ the linear utility

$$g^{(t)}(h) := \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h\left(\boldsymbol{c}^{(t)}, \boldsymbol{p}^{(t)}\right), \boldsymbol{u}^{(t)} - \boldsymbol{p}^{(t)} \right\rangle \right].$$

[1] Expected variational inequalities have appeared in the literature under different names, including "outgoing minimax problems" [FH21], "accuracy certificates" [NOR10], or "negative correlation search" [PR25].

Theorem 4.2 ([FP26]). Let $\mathrm{Reg}_{\mathcal{H}}^{(T)}(h)$ be the external regret of $R_{\mathcal{H}}$ with respect to the sequence of utilities $g^{(1)}, ..., g^{(T)}$, and let $\mathrm{EVI}^{(T)} := \sum_{t=1}^{T} \varepsilon^{(t)}$. Then the forecasts produced by the algorithm above satisfy

$$\text{MC-Err}^{(T)}(h) \le \mathrm{Reg}_{\mathcal{H}}^{(T)}(h) + \mathrm{EVI}^{(T)} \quad \forall h \in \mathcal{H}.$$

Proof. The EVI condition is invoked at the realized utility vector $\boldsymbol{u}^{(t)}$, so

$$g^{(t)}\big(h^{(t)}\big) \le \varepsilon^{(t)}.$$

Therefore, for any comparator test $h \in \mathcal{H}$,

$$\begin{aligned}\text{MC-Err}^{(T)}(h) &= \sum_{t=1}^{T} g^{(t)}(h) \\ &\le \sum_{t=1}^{T} \big(g^{(t)}(h) - g^{(t)}\big(h^{(t)}\big)\big) + \sum_{t=1}^{T} \varepsilon^{(t)} \\ &= \mathrm{Reg}_{\mathcal{H}}^{(T)}(h) + \mathrm{EVI}^{(T)}.\end{aligned}$$

Thus, sublinear regret over $\mathcal{H}$, together with $\sum^{(T)} \varepsilon^{(t)} = o(T)$, gives sublinear multicalibration error. □

Necessity of EVIs. In a precise sense, EVIs are also *necessary*. Suppose we already had an efficient $\mathcal{H}$-multicalibrated forecaster. Fix a context $\boldsymbol{c} \in \mathcal{C}$ and a test $h \in \mathcal{H}$, and consider the EVI operator $S(\boldsymbol{p}) = h(\boldsymbol{c}, \boldsymbol{p})$. To solve this EVI, run the forecaster for T rounds with the same context $\boldsymbol{c}^{(t)} = \boldsymbol{c}$. After it outputs $D^{(t)}$, choose Nature's response as

$$\boldsymbol{u}^{(t)} \in \arg\max_{\boldsymbol{u} \in \mathcal{U}} \left\langle \boldsymbol{u}, \mathop{\mathbb{E}}_{\boldsymbol{p} \sim D^{(t)}} [S(\boldsymbol{p})] \right\rangle. \tag{3}$$

For any fixed $\boldsymbol{u}^\star \in \mathcal{U}$, the choice in (3) gives

$$\mathop{\mathbb{E}}_{\boldsymbol{p} \sim D^{(t)}} \big[\big\langle S(\boldsymbol{p}), \boldsymbol{u}^{(t)} - \boldsymbol{p}\big\rangle\big] \ge \mathop{\mathbb{E}}_{\boldsymbol{p} \sim D^{(t)}} [\langle S(\boldsymbol{p}), \boldsymbol{u}^\star - \boldsymbol{p}\rangle].$$

Averaging the distributions $D^{(1)}, ..., D^{(T)}$ uniformly therefore produces a distribution D with

$$\mathop{\mathbb{E}}_{\boldsymbol{p} \sim D} [\langle S(\boldsymbol{p}), \boldsymbol{u}^\star - \boldsymbol{p}\rangle] \le \frac{\text{MC-Err}^{(T)}(h)}{T} \quad \forall \boldsymbol{u}^\star \in \mathcal{U}.$$

As T grows, the multicalibration condition drives the right-hand side to 0. So an online multicalibrated forecaster gives a black-box EVI solver.

4.2 From Multicalibration to Phi-regret minimization

We now use forecasting to construct a Φ-regret minimizer. Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a compact convex action set, $\mathcal{U} \subseteq \mathbb{R}^d$ be a compact convex set of possible utility vectors, and $\Phi \subseteq \{\varphi : \mathcal{X} \to$

$\mathcal{X}\}$ be a family of deviations. Recall that if the learner outputs distributions $\mu^{(t)}$ over $\mathcal{X}$, its Φ-regret against a deviation $\varphi \in \Phi$ is

$$\Phi\mathrm{Reg}^{(T)}(\varphi) := \sum_{t=1}^{T} \mathop{\mathbb{E}}_{\boldsymbol{x}^{(t)} \sim \mu^{(t)}} \left[\left\langle \varphi\left(\boldsymbol{x}^{(t)}\right) - \boldsymbol{x}^{(t)}, \boldsymbol{u}^{(t)} \right\rangle\right]. \tag{4}$$

The idea of the reduction asks the forecaster to predict the next utility vector. Given a forecast $\boldsymbol{p} \in \mathcal{U}$, define the deterministic best response

$$\sigma(\boldsymbol{p}) \in \arg\max_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{x}, \boldsymbol{p} \rangle, \tag{5}$$

with ties broken by a fixed rule. If the forecaster outputs a distribution $D^{(t)}$ over forecasts $\boldsymbol{p}^{(t)}$, the decision maker plays the pushforward distribution $\mu^{(t)}$ induced by $\boldsymbol{x}^{(t)} = \sigma\left(\boldsymbol{p}^{(t)}\right)$.

What tests should the forecaster be calibrated against? For each deviation $\varphi \in \Phi$, define

$$h_{\varphi}(\boldsymbol{p}) := \varphi(\sigma(\boldsymbol{p})) - \sigma(\boldsymbol{p}), \qquad \mathcal{H}_{\Phi} := \left\{h_{\varphi} : \varphi \in \Phi\right\}. \tag{6}$$

As it turns out, multicalibration with respect to the test class $\mathcal{H}_{\Phi}$ suffices to control Φ-regret.

Algorithm: Φ-regret from multicalibration

Input: A forecaster over $\mathcal{U}$ multicalibrated with respect to $\mathcal{H}_{\Phi}$

`NextStrategy`():

- Query the forecaster and receive a distribution $D^{(t)}$ over forecasts $\boldsymbol{p}^{(t)} \in \mathcal{U}$
- **return** the pushforward distribution $\mu^{(t)}$ of $\boldsymbol{x}^{(t)} = \sigma\left(\boldsymbol{p}^{(t)}\right)$

`ObserveUtility`($\boldsymbol{u}^{(t)}$):

- Feed the realized utility vector $\boldsymbol{u}^{(t)}$ to the forecaster

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Theorem 4.3 ([FP26]). If the forecaster in the algorithm above has multicalibration error $\text{MC-Err}^{(T)}$ with respect to $\mathcal{H}_{\Phi}$, then the decision maker has

$$\Phi\mathrm{Reg}^{(T)}(\varphi) \leq \text{MC-Err}^{(T)}\left(h_{\varphi}\right) \quad \forall \varphi \in \Phi.$$

In particular, sublinear $\mathcal{H}_{\Phi}$-multicalibration implies sublinear Φ-regret.

Proof. Fix any $\varphi \in \Phi$. Since $\mu^{(t)}$ is the pushforward of $D^{(t)}$ under $\boldsymbol{p} \mapsto \sigma(\boldsymbol{p})$,

$$\begin{aligned}
\Phi\mathrm{Reg}^{(T)}(\varphi) &= \sum_{t=1}^{T} \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h_{\varphi}\left(\boldsymbol{p}^{(t)}\right), \boldsymbol{u}^{(t)} \right\rangle\right] \\
&= \sum_{t=1}^{T} \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h_{\varphi}\left(\boldsymbol{p}^{(t)}\right), \boldsymbol{u}^{(t)} - \boldsymbol{p}^{(t)} \right\rangle\right] + \sum_{t=1}^{T} \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)} \sim D^{(t)}} \left[\left\langle h_{\varphi}\left(\boldsymbol{p}^{(t)}\right), \boldsymbol{p}^{(t)} \right\rangle\right].
\end{aligned}$$

The first term is exactly $\text{MC-Err}^{(T)}(h_{\varphi})$. The second term is nonpositive, because

$$\left\langle h_{\varphi}(\boldsymbol{p}), \boldsymbol{p} \right\rangle = \langle \varphi(\sigma(\boldsymbol{p})), \boldsymbol{p} \rangle - \langle \sigma(\boldsymbol{p}), \boldsymbol{p} \rangle \leq 0$$

by the definition of $\sigma(\boldsymbol{p})$ as a maximizer over $\mathcal{X}$. □

Quite intuitively, the complexity of the required calibration class scales with the complexity of the deviation class. We can think of Theorem 4.3 as a generalization of the classical connection between ℓ_1-calibration and swap regret [FV97].

Remark 4.4. We can equally easily define a contextual version of the result. If contexts $\boldsymbol{c}^{(t)}$ are observed before play and deviations have the form $\varphi : \mathcal{C} \times \mathcal{X} \to \mathcal{X}$, then we define

$$h_\varphi(\boldsymbol{c}, \boldsymbol{p}) := \varphi(\boldsymbol{c}, \sigma(\boldsymbol{p})) - \sigma(\boldsymbol{p}).$$

Multicalibration with respect to these tests gives a notion of *contextual* Φ-regret.

4.2.1 Putting the two reductions together

Combining Theorem 4.2 and Theorem 4.3 gives the promised route from external regret to Φ-regret. Set $\mathcal{H} = \mathcal{H}_\Phi$. Run the multicalibration algorithm as the forecaster inside the best-response algorithm. Then, for every $\varphi \in \Phi$,

$$\Phi\mathrm{Reg}^{(T)}(\varphi) \le \text{MC-Err}^{(T)}(h_\varphi) \le \mathrm{Reg}_{\mathcal{H}}^{(T)}(h_\varphi) + \mathrm{EVI}^{(T)}. \tag{7}$$

Thus, the burden of Φ-regret minimization shifts to two primitives:

1. no-regret learning over the induced test class $\mathcal{H}_\Phi$; and
2. solving EVIs over the forecast domain $\mathcal{U}$.

The key distinction from the Gordon-Greenwald-Marks path is that we do *not* need to optimize over valid deviations directly. We only need a regret minimizer over tests that contain the maps h_φ. This extra flexibility is useful when characterizing the deviation class would otherwise require complex machinery, such the semiseparation construction of Daskalakis, Farina, Fishelson, Pipis and Schneider [DFF+25] mentioned in Chapter 2, Section 2.2.

4.2.2 Linear deviations

As an illustration, consider linear deviations $\varphi(\boldsymbol{x}) = \mathbf{M}\boldsymbol{x}$ over a convex compact set $\mathcal{X} \subseteq \mathbb{R}^d$. The classical GGM approach asks us to understand the geometry of all matrices $\mathbf{M}$ satisfying $\mathbf{M}\mathcal{X} \subseteq \mathcal{X}$, and then compute fixed points of the matrices selected by the regret minimizer. That geometry of the endomorphisms can be difficult [DFF+25].

The multicalibration route permits a relaxation. From (6),

$$h_\varphi(\boldsymbol{p}) = (\mathbf{M} - \mathbf{I})\sigma(\boldsymbol{p}).$$

If every valid linear endomorphism has spectral norm at most S, then each test above lies in a Frobenius ball of radius on the order of $\sqrt{d}(S+1)$. Hence, instead of learning over valid endomorphisms, we can learn over the larger class

$$\mathcal{H} := \{\boldsymbol{p} \mapsto \mathbf{A}\sigma(\boldsymbol{p}) : \|\mathbf{A}\|_F \le \rho\},$$

where ρ is chosen large enough to contain all tests h_φ. This larger class is easy: external regret over a Euclidean ball can be easily handled by projected online gradient descent. Indeed, recall that the utility sent to the matrix learner at time t is linear:

$$g^{(t)}(\mathbf{A}) = \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)}\sim D^{(t)}}\left[\left\langle \mathbf{A}\sigma\left(\boldsymbol{p}^{(t)}\right), \boldsymbol{u}^{(t)} - \boldsymbol{p}^{(t)}\right\rangle\right] = \left\langle \mathbf{A}, \mathop{\mathbb{E}}_{\boldsymbol{p}^{(t)}\sim D^{(t)}}\left[\left(\boldsymbol{u}^{(t)} - \boldsymbol{p}^{(t)}\right)\sigma\left(\boldsymbol{p}^{(t)}\right)^\top\right]\right\rangle_F.$$

Consequently, if $\|\boldsymbol{x}\|_2 \le B$ for $\boldsymbol{x} \in \mathcal{X}$ and $\|\boldsymbol{u}\|_2 \le L$ for $\boldsymbol{u} \in \mathcal{U}$, the standard projected-gradient bound gives $O\left(BLS\sqrt{dT}\right)$ linear-swap regret, up to constants. This improves the dimension dependence of the recent semi-separation-based route of Daskalakis, Farina, Fishelson, Pipis and Schneider [DFF+25], while avoiding semi-separation altogether.

4.2.3 RKHS deviations

The same idea extends beyond finite-dimensional linear classes. Suppose that the deviation functions $\varphi : \mathcal{C} \times \mathcal{X} \to \mathcal{X}$ lie in a vector-valued reproducing kernel Hilbert space with matrix-valued kernel

$$\Gamma((\boldsymbol{c}, \boldsymbol{x}), (\boldsymbol{c}', \boldsymbol{x}')) \in \mathbb{R}^{d\times d}.$$

For the contextual reduction, the relevant tests are

$$h_\varphi(\boldsymbol{c}, \boldsymbol{p}) = \varphi(\boldsymbol{c}, \sigma(\boldsymbol{p})) - \sigma(\boldsymbol{p}).$$

These tests also lie in an RKHS. Indeed, the first term is represented by composing the deviation kernel with the best-response map, while the second term is represented by the linear kernel on $\sigma(\boldsymbol{p})$. The resulting kernel is

$$\Gamma'((\boldsymbol{c}, \boldsymbol{p}), (\boldsymbol{c}', \boldsymbol{p}')) := \Gamma((\boldsymbol{c}, \sigma(\boldsymbol{p})), (\boldsymbol{c}', \sigma(\boldsymbol{p}'))) + \sigma(\boldsymbol{p})\sigma(\boldsymbol{p}')^\top. \tag{8}$$

So, if we have an online learner over the RKHS ball associated with Γ', Theorem 4.2 supplies a multicalibrated forecaster, and Theorem 4.3 turns it into a no-Φ-regret algorithm. This recovers low-degree polynomial deviations as a special case and also covers infinite-dimensional kernels, such as Gaussian kernels, where the fixed-point route does not have an obvious finite-dimensional endomorphism geometry to exploit.

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